Linear Algebra I

13/04/2022, Wednesday, 18:45 – 20:45

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1 Linear equations (15 + 5 = 20 pts)

Consider the linear equation

$$\begin{bmatrix} 1 & 3 & 5 & -2 \\ 1 & 4 & 6 & -2 \\ -1 & -1 & -3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ b-2a \end{bmatrix}$$

where a and b are real numbers.

a. Find all values of a and b for which the equation is consistent. For these values find the general solution of the equation.

b. Find all values of a and b for which the equation has a unique solution.

2 Vector spaces (3 + 4 + 3 + 6 + 6 + 8 = 30 pts)

Consider the vector space P_4 (the set of polynomials of degree less than 4). Let $S \subseteq P_4$ be defined as

$$S := \{p(x) \in P_4 \mid p(x) = x^3 p(\frac{1}{x})\}.$$

a. Show that S is a subspace of P_4.

b. Find a basis for S.

c. Find the dimension of S.

Let $T : P_4 \to P_4$ be defined as

$$T(p(x)) := p(x) + x^2 p'(\frac{1}{x})$$

where $p'(x)$ denotes the derivative of $p(x)$.

d. Show that T is a linear transformation.

e. Determine $\ker(T)$.

f. Find the matrix representation of T with respect to the ordered basis $\{1, x, x^2, x^3\}$.

3 Partitioned matrices and nonsingularity (10 + 10 = 20 pts)

Let $M \in \mathbb{R}^{p \times q}$ and consider the partitioned matrix

$$N = \begin{bmatrix} I_p & M \\ M^T & I_q \end{bmatrix}.$$

a. Show that N is nonsingular if and only if 1 is not an eigenvalue of MM^T.

b. Suppose that N is nonsingular. Find N^{-1}.

4 Determinants, eigenvalues, and diagonalization (2 + 2 + 4 + 6 + 6 = 20 pts)

Consider the matrix

$$M = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 3 & -1 \\ 0 & 0 & \alpha \end{bmatrix}$$

where α is a real number.

a. Find the determinant of M.

b. Find all values of α for which M is nonsingular.

c. Find the eigenvalues of M.

d. Find all values of α for which M is diagonalizable.

e. Let $\alpha = 1$. Find a nonsingular matrix T and a diagonal matrix D such that $M = TDT^{-1}$.

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10 pts free